CARDINAL SEQUENCES OF LCS SPACES UNDER GCH

JUAN CARLOS MARTINEZ AND LAJOS SOUKUP

ABSTRACT. Let $\mathcal{C}(\alpha)$ denote the class of all cardinal sequences of length α associated with compact scattered spaces. Also put

$$\mathcal{C}_\lambda(\alpha) = \{f \in \mathcal{C}(\alpha) : f(0) = \lambda = \min[f(\beta) : \beta < \alpha]\}.$$

If λ is a cardinal and $\alpha < \lambda^{++}$ is an ordinal, we define $\mathcal{D}_\lambda(\alpha)$ as follows: if $\lambda = \omega$,

$$\mathcal{D}_\omega(\alpha) = \{f \in {}^\alpha\{\omega, \omega_1\} : f(0) = \omega\},$$

and if λ is uncountable,

$$\mathcal{D}_\lambda(\alpha) = \{f \in {}^\alpha\{\lambda, \lambda^+\} : f(0) = \lambda,$$
$$f^{-1}\{\lambda\} \text{ is } < \lambda\text{-closed and successor-closed in } \alpha\}.$$

We show that for each uncountable regular cardinal λ and ordinal $\alpha < \lambda^{++}$ it is consistent with GCH that $\mathcal{C}_\lambda(\alpha)$ is as large as possible, i.e.

$$\mathcal{C}_\lambda(\alpha) = \mathcal{D}_\lambda(\alpha).$$

This yields that under GCH for any sequence f of regular cardinals of length α the following statements are equivalent:

(1) $f \in \mathcal{C}(\alpha)$ in some cardinal preserving and GCH-preserving generic-extension of the ground model.
(2) for some natural number n there are infinite regular cardinals $\lambda_0 > \lambda_1 > \cdots > \lambda_{n-1}$ and ordinals $\alpha_0, \ldots, \alpha_{n-1}$ such that $\alpha = \alpha_0 + \cdots + \alpha_{n-1}$ and $f = f_0 {}^\frown f_1 {}^\frown \cdots {}^\frown f_{n-1}$ where each $f_i \in \mathcal{D}_{\lambda_i}(\alpha_i)$.

The proofs are based on constructions of *universal* locally compact scattered spaces.

2000 *Mathematics Subject Classification.* 54A25, 06E05, 54G12, 03E35, 03E05.

Key words and phrases. locally compact scattered space, superatomic Boolean algebra, cardinal sequence, universal.

The first author was supported by the Spanish Ministry of Education DGI grant MTM2005-00203.

The second author was partially supported by Hungarian National Foundation for Scientific Research grant no 61600. The research was started when the second author visited the Barcelona University. The second author would like to thank Joan Bagaria and Juan-Carlos Martínez for the arrangement of the visit and their hospitality during the stay in Barcelona.

1. Introduction

Given a locally compact scattered T_2 (in short : LCS) space X the α^{th} Cantor-Bendixson level will be denoted by $\mathrm{I}_\alpha(X)$. The *height of* X, $\mathrm{ht}(X)$, is the least ordinal α with $\mathrm{I}_\alpha(X) = \emptyset$. The *reduced height* $\mathrm{ht}^-(X)$ is the smallest ordinal α such that $\mathrm{I}_\alpha(X)$ is finite. Clearly, one has $\mathrm{ht}^-(X) \le \mathrm{ht}(X) \le \mathrm{ht}^-(X) + 1$. The *cardinal sequence* of X, denoted by $\mathrm{SEQ}(X)$, is the sequence of cardinalities of the infinite Cantor-Bendixson levels of X, i.e.

$$\mathrm{SEQ}(X) = \left\langle\ |I_\alpha(X)| : \alpha < \mathrm{ht}(X)^-\ \right\rangle.$$

A characterization in ZFC of the sequences of cardinals of length $\le \omega_1$ that arise as cardinal sequences of LCS spaces is proved in [4]. However, no characterization in ZFC is known for cardinal sequences of length $< \omega_2$.

For an ordinal α we let $\mathcal{C}(\alpha)$ denote the class of all cardinal sequences of length α of LCS spaces. We also put, for any fixed infinite cardinal λ,

$$\mathcal{C}_\lambda(\alpha) = \{s \in \mathcal{C}(\alpha) : s(0) = \lambda \wedge \forall \beta < \alpha\ [s(\beta) \ge \lambda]\}.$$

In [2], the authors show that a class $\mathcal{C}(\alpha)$ is characterized if the classes $\mathcal{C}_\lambda(\beta)$ are characterized for every infinite cardinal λ and every ordinal $\beta \le \alpha$. Then, they obtain under GCH a characterization of the classes $\mathcal{C}(\alpha)$ for any ordinal $\alpha < \omega_2$ by means of a a full description under GCH of the classes $\mathcal{C}_\lambda(\alpha)$ for any ordinal $\alpha < \omega_2$ and any infinite cardinal λ. The situation becomes, however, more complicated when we consider the class $\mathcal{C}(\omega_2)$. We can characterize under GCH the classes $\mathcal{C}_\lambda(\omega_2)$ for $\lambda > \omega_1$, by using the description given in [2] and the following simple observation.

Observation 1.1. *If $\lambda \ge \omega_2$, then $f \in \mathcal{C}_\lambda(\omega_2)$ iff $f \restriction \alpha \in \mathcal{C}_\lambda(\alpha)$ for each $\alpha < \omega_2$.*

Proof. If $\mathrm{SEQ}(X_\alpha) = f \restriction \alpha$ for $\alpha < \omega_2$ then take X as the disjoint union of $\{X_\alpha : \alpha < \omega_2\}$. Then $\mathrm{SEQ}(X) = f$ because for any $\beta < \omega_2$ we have $\mathrm{I}_\beta(X) = \bigcup\{\mathrm{I}_\beta(X_\alpha) : \beta < \alpha < \omega_2\}$ and so

$$|\,\mathrm{I}_\beta(X)| = \sum_{\beta<\alpha<\omega_2} |\,\mathrm{I}_\beta(X_\alpha)| = \omega_2 \cdot f(\beta) = f(\beta).$$

□

If α is any ordinal, a subset $L \subset \alpha$ is called *κ-closed in α*, where κ is an infinite cardinal, iff $\sup \langle \alpha_i : i < \kappa\rangle \in L \cup \{\alpha\}$ for each increasing sequence $\langle \alpha_i : i < \kappa\rangle \in {}^\kappa L$. The set L is *$< \lambda$-closed in α* provided it

is κ-closed in α for each cardinal $\kappa < \lambda$. We say that L is *successor closed in* α if $\beta + 1 \in L \cup \{\alpha\}$ for all $\beta \in L$.

For a cardinal λ and ordinal $\delta < \lambda^{++}$ we define $\mathcal{D}_\lambda(\delta)$ as follows: if $\lambda = \omega$,

$$\mathcal{D}_\omega(\delta) = \{f \in {}^\delta\{\omega, \omega_1\} : f(0) = \omega\},$$

and if λ is uncountable,

$$\begin{aligned}\mathcal{D}_\lambda(\delta) = \{s \in {}^\delta\{\lambda, \lambda^+\} : s(0) = \lambda,& \\ s^{-1}\{\lambda\} \text{ is } < \lambda\text{-closed and successor-closed in } \delta\}.&\end{aligned}$$

The observation 1.1 above left open the characterization of $\mathcal{C}_{\omega_1}(\omega_2)$ under GCH. In [2, Theorem 4.1] it was proved that if GCH holds then

$$\mathcal{C}_{\omega_1}(\delta) \subseteq \mathcal{D}_{\omega_1}(\delta),$$

and we have equality for $\delta < \omega_2$. In Theorem 1.3 we show that it is consistent with GCH that we have equality not only for $\delta = \omega_2$ but even for each $\delta < \omega_3$.

To formulate our results we need to introduce some more notation.

We shall use the notation $\langle\kappa\rangle_\alpha$ to denote the constant κ-valued sequence of length α. Let us denote the concatenation of a sequence f of length α and a sequence g of length β by $f^\frown g$ so that the domain of $f^\frown g$ is $\alpha + \beta$ and $f^\frown g(\xi) = f(\xi)$ for $\xi < \alpha$ and $f^\frown g(\alpha + \xi) = g(\xi)$ for $\xi < \beta$.

Definition 1.2. An LCS space X is called $\mathcal{C}_\lambda(\alpha)$-*universal* iff $\mathrm{SEQ}(X) \in \mathcal{C}_\lambda(\alpha)$ and for each sequence $s \in \mathcal{C}_\lambda(\alpha)$ there is an open subspace Y of X with $\mathrm{SEQ}(Y) = s$.

In this paper we prove the following result:

Theorem 1.3. *If κ is an uncountable regular cardinal with $\kappa^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$ then for each $\delta < \kappa^{++}$ there is a κ-complete κ^+-c.c poset P of cardinality κ^+ such that in V^P*

$$\mathcal{C}_\kappa(\delta) = \mathcal{D}_\kappa(\delta)$$

and there is a $\mathcal{C}_\kappa(\delta)$-universal LCS space.

How do the universal spaces come into the picture? The first idea to prove the consistency of $\mathcal{C}_\lambda(\alpha) = \mathcal{D}_\lambda(\alpha)$ is to try to carry out an iterated forcing. For each $f \in \mathcal{D}_\lambda(\alpha)$ we can try to find a poset P_f such that

$$1_{P_f} \Vdash \text{There is an LCS space } X_f \text{ with cardinal sequence } f.$$

Since typically $|X_f| = \lambda^+$, if we want to preserve the cardinals and CGH we should try to find a λ-complete, λ^+-c.c. poset P_f of cardinality λ^+. In this case forcing with P_f introduces λ^+ new subsets of λ because P_f has cardinality λ^+. However $|\mathcal{D}_\lambda(\alpha)| = \lambda^{++}$! So the length of the iteration is at least λ^{++}, hence in the final model the cardinal λ will have $\lambda^+ \cdot \lambda^{++} = \lambda^{++}$ many new subsets, i.e. $2^\lambda > \lambda^+$.

A $\mathcal{C}_\lambda(\delta)$-universal space has cardinality λ^+ so we may hope that there is a λ-complete, λ^+-c.c. poset P of cardinality λ^+ such that V^P contains a $\mathcal{C}_\lambda(\delta)$-universal space. In this case $(2^\lambda)^{V^P} \le ((|P|^\lambda)^\lambda)^V = \lambda^+$. So in the generic extension we might have GCH.

In this paper, we shall use the notion of a universal LCS space in order to prove Theorem 1.3. Further constructions of universal LCS spaces will be carried out in [6].

Problem 1.4. *Assume that s is a sequence of cardinals of length α, $s \notin \mathcal{C}(\alpha)$. Is it possible that there is a $|\alpha|^+$-Baire ($|\alpha|^+$-complete) poset P such that $s \in \mathcal{C}(\alpha)$ in V^P?*

For an ordinal $\delta < \kappa^{++}$ let $\mathcal{L}^\delta_\kappa = \{\alpha < \delta : \mathrm{cf}(\alpha) \in \{\kappa, \kappa^+\}\}$.

Definition 1.5. An LCS space X is called $\mathcal{L}^\delta_\kappa$-good iff X has a partition $X = Y \cup^* \bigcup^* \{Y_\zeta : \zeta \in \mathcal{L}^\delta_\kappa\}$ such that

(1) Y is an open subspace of X, $\mathrm{SEQ}(Y) = \langle\kappa\rangle_\delta$,
(2) $Y \cup Y_\zeta$ is an open subspace of X with $\mathrm{SEQ}(Y \cup Y_\zeta) = \langle\kappa\rangle_\zeta {}^\frown \langle\kappa^+\rangle_{\delta-\zeta}$.

Theorem 1.3 follows immediately from Theorem 1.6 and Proposition 1.7 below.

Theorem 1.6. *If κ is an uncountable regular cardinal with $\kappa^{<\kappa} = \kappa$ then for each $\delta < \kappa^{++}$ there is a κ-complete κ^+-c.c poset $\mathcal{P}$ of cardinality κ^+ such that in $V^{\mathcal{P}}$ there is an $\mathcal{L}^\delta_\kappa$-good space.*

Proposition 1.7. *Let κ be an uncountable regular cardinal, $\delta < \kappa^{++}$ and X be an $\mathcal{L}^\delta_\kappa$-good space. Then for each $s \in \mathcal{D}_\kappa(\delta)$ there is an open subspace Z of X with $\mathrm{SEQ}(Z) = s$. Especially, under GCH an $\mathcal{L}^\delta_\kappa$-good space is $\mathcal{C}_\kappa(\delta)$-universal.*

Proof. Let $J = s^{-1}\{\kappa^+\} \cap \mathcal{L}^\delta_\kappa$. For each $\zeta \in J$ let

$$f(\zeta) = \min((\delta + 1) \setminus (s^{-1}\{\kappa^+\} \cup \zeta)).$$

Let

$$Z = Y \cup \bigcup\{\mathrm{I}_{<f(\zeta)}(Y \cup Y_\zeta) : \zeta \in J\}.$$

Since $Y \cup Y_\zeta$ is an open subspace of X it follows that $\mathrm{I}_{<f(\zeta)}(Y \cup Y_\zeta)$ is an open subspace of Z. Hence for every $\alpha < \delta$

$$
\begin{aligned}
(1) \quad \mathrm{I}_\alpha(Z) &= \mathrm{I}_\alpha(Y) \cup \bigcup\{\mathrm{I}_\alpha(\mathrm{I}_{<f(\zeta)}(Y \cup Y_\zeta)) : \zeta \in J\} \\
&= \mathrm{I}_\alpha(Y) \cup \bigcup\{\mathrm{I}_\alpha(Y \cup Y_\zeta) : \zeta \in J, \zeta \le \alpha < f(\zeta)\}.
\end{aligned}
$$

Since $[\zeta, f(\zeta)) \subset s^{-1}\{\kappa^+\}$ for $\zeta \in J$ it follows that if $s(\alpha) = \kappa$ then $\mathrm{I}_\alpha(Z) = \mathrm{I}_\alpha(Y)$, and so

$$(2) \qquad |\,\mathrm{I}_\alpha(Z)| = |\,\mathrm{I}_\alpha(Y)| = \kappa.$$

If $s(\alpha) = \kappa^+$, let $\zeta_\alpha = \min\{\zeta \le \alpha : [\zeta, \alpha] \subset s^{-1}\{\kappa^+\}\}$. Then $\zeta_\alpha \in J$ because $s(0) = \kappa$ and $s^{-1}\{\kappa\}$ is $< \kappa$-closed and successor-closed in δ. Thus $\zeta_\alpha \le \alpha < f(\zeta_\alpha)$ and so

$$(3) \qquad |\,\mathrm{I}_\alpha(Z)| \ge |\,\mathrm{I}_\alpha(Y \cup Y_{\zeta_\alpha})| = \kappa^+.$$

Since $|Z| \le |X| = \kappa^+$ we have $|\,\mathrm{I}_\alpha(Z)| = \kappa^+$. Thus $\mathrm{SEQ}(Z) = s$. $\square$

Theorem 1.3 yields the following characterization:

Theorem 1.8. *Under GCH for any sequence f of regular cardinals of length α the following statements are equivalent:*

(A) $f \in \mathcal{C}(\alpha)$ in some cardinal preserving and GCH-preserving generic-extension of the ground model.

(B) for some natural number n there are infinite regular cardinals $\lambda_0 > \lambda_1 > \cdots > \lambda_{n-1}$ and ordinals $\alpha_0, \ldots, \alpha_{n-1}$ such that $\alpha = \alpha_0 + \cdots + \alpha_{n-1}$ and $f = f_0{}^\frown f_1{}^\frown \cdots {}^\frown f_{n-1}$ where each $f_i \in \mathcal{D}_{\lambda_i}(\alpha_i)$.

Proof. (A) clearly implies (B) by [2].

Assume now that (B) holds. Without loss of generality, we may suppose that $\lambda_{n-1} = \omega$. Since the notion of forcing defined in Theorem 1.3 preserves GCH, we can carry out a cardinal-preserving and GCH-preserving iterated forcing of length $n-1$, $\langle P_m : m < n-1\rangle$, such that for $m < n-1$

$$V^{P_m} \models \mathcal{C}_{\lambda_m}(\alpha_m) = \mathcal{D}_{\lambda_m}(\alpha_m).$$

Put $k = n-2$, $\beta = \alpha_0 + \cdots + \alpha_k$ and $g = f_0{}^\frown f_1{}^\frown \cdots {}^\frown f_k$. Since $f_m \in \mathcal{D}_{\lambda_m}(\alpha_m) \cap V$, in V^{P_k} we have $f_m \in \mathcal{C}_{\lambda_m}(\alpha_m)$ for each $m < n-1$. Hence in V^{P_k} we have $g \in \mathcal{C}(\beta)$ by [2, Lemma 2.2]. Also, by using [4, Theorem 9], we infer that $f_{n-1} \in \mathcal{C}(\alpha_{n-1})$ in ZFC. Then as $f = g^\frown f_{n-1}$, in V^{P_k} we have $f \in \mathcal{C}(\alpha)$ again by [2, Lemma 2.2]. $\square$

Problem 1.9. (1) *Are (A) and (B) below equivalent under GCH for every sequence f of regular cardinals?*

(A) $f \in \mathcal{C}(\alpha)$.

(B) for some natural number n there are infinite regular cardinals $\lambda_0 > \lambda_1 > \cdots > \lambda_{n-1}$ *and ordinals* $\alpha_0, \ldots, \alpha_{n-1}$ *such that* $\alpha = \alpha_0 + \cdots + \alpha_{n-1}$ *and* $f = f_0{}^\frown f_1{}^\frown \cdots {}^\frown f_{n-1}$ *where each* $f_i \in \mathcal{D}_{\lambda_i}(\alpha_i)$*.*

(2) *Is it consistent with GCH that (A) and (B) above are equivalent for every sequence of regular cardinals?*

Juhász and Weiss proved in [3] that $\langle\omega\rangle_\delta \in \mathcal{C}(\delta)$ for each $\delta < \omega_2$.

Also, it was shown in [5] that for every specific regular cardinal κ it is consistent that $\langle\kappa\rangle_\delta \in \mathcal{C}(\delta)$ for each $\delta < \kappa^{++}$. However, the following problem is open:

Problem 1.10. *Is it consistent with GCH that* $\langle\omega_1\rangle_\delta \in \mathcal{C}(\delta)$ *for each* $\delta < \omega_3$*?*

2. Proof of theorem 1.6

This section is devoted to the proof of Theorem 1.6, so κ is an uncountable regular cardinal with $\kappa^{<\kappa} = \kappa$, and $\delta < \kappa^{++}$ is an ordinal.

If $\alpha \le \beta$ are ordinals let

$$[\alpha, \beta) = \{\gamma : \alpha \le \gamma < \beta\}. \tag{4}$$

We say that I is an *ordinal interval* iff there are ordinals α and β with $I = [\alpha, \beta)$. Write $I^- = \alpha$ and $I^+ = \beta$.

If $I = [\alpha, \beta)$ is an ordinal interval let $\mathrm{E}(I) = \{\varepsilon^I_\nu : \nu < \mathrm{cf}(\beta)\}$ be a cofinal closed subset of I having order type $\mathrm{cf}\,\beta$ with $\alpha = \varepsilon^I_0$ and put

$$\mathcal{E}(I) = \{[\varepsilon^I_\nu, \varepsilon^I_{\nu+1}) : \nu < \mathrm{cf}\,\beta\} \tag{5}$$

provided β is a limit ordinal, and let $\mathrm{E}(I) = \{\alpha, \beta'\}$ and put

$$\mathcal{E}(I) = \{[\alpha, \beta'), \{\beta'\}\} \tag{6}$$

provided $\beta = \beta' + 1$.

Define $\{\mathcal{I}_n : n < \omega\}$ as follows:

$$\mathcal{I}_0 = \{[0, \delta)\} \text{ and } \mathcal{I}_{n+1} = \bigcup\{\mathcal{E}(I) : I \in \mathcal{I}_n\}. \tag{7}$$

Put $\mathbb{I} = \bigcup\{\mathcal{I}_n : n < \omega\}$. Note that $\mathbb{I}$ is a *cofinal tree of intervals* in the sense defined in [5]. Then, for each $\alpha < \delta$ we define

$$\mathrm{n}(\alpha) = \min\{n : \exists I \in \mathcal{I}_n \text{ with } I^- = \alpha\}, \tag{8}$$

and for each $\alpha < \delta$ and $n < \omega$ we define

$$\mathrm{I}(\alpha, n) \in \mathcal{I}_n \text{ such that } \alpha \in \mathrm{I}(\alpha, n). \tag{9}$$

Proposition 2.1. *Assume that* $\zeta < \delta$ *is a limit ordinal. Then, there is a* $j(\zeta) \in \omega$ *and an interval* $J(\zeta) \in \mathcal{I}_{j(\zeta)}$ *such that* ζ *is a limit point of* $E(J(\zeta))$*. Also, we have* $\mathrm{n}(\zeta) - 1 \le j(\zeta) \le \mathrm{n}(\zeta)$*, and* $j(\zeta) = n(\zeta)$ *if* $cf(\zeta) = \kappa^+$*.*

Proof. Clearly $j(\zeta)$ and $\mathrm{J}(\zeta)$ are unique if defined.

If there is an $I \in \mathcal{I}_{\mathrm{n}(\zeta)}$ with $I^+ = \zeta$ then $J(\zeta) = I$, and so $j(\zeta) = \mathrm{n}(\zeta)$. If there is no such I, then ζ is a limit point of $\mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1))$, so $J(\zeta) = \mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1)$ and $j(\zeta) = \mathrm{n}(\zeta) - 1$.

Assume now that $\mathrm{cf}(\zeta) = \kappa^+$. Then $\zeta \in \mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1))$, but $|\,\mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1)) \cap \zeta| \leq \kappa$, so ζ can not be a limit point of $\mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1))$. Therefore, it has a predecessor ξ in $\mathrm{E}(\mathrm{I}(\zeta, \mathrm{n}(\zeta) - 1))$, i.e $[\xi, \zeta) \in \mathcal{I}_{\mathrm{n}(\zeta)}$, and so $J(\zeta) = [\xi, \zeta)$ and $j(\zeta) = \mathrm{n}(\zeta)$. □

Example 2.2. Put $\delta = \omega_2 \cdot \omega_2 + 1$. We define

$E([0, \delta)) = \{0, \omega_2 \cdot \omega_2\}$,

$E([0, \omega_2 \cdot \omega_2)) = \{\omega_2 \cdot \xi : 0 \leq \xi < \omega_2\}$,

$E([\omega_2 \cdot \xi, \omega_2 \cdot (\xi + 1))) = \{\zeta : \omega_2 \cdot \xi \leq \zeta < \omega_2 \cdot (\xi + 1)\}$,

$E(\{\zeta\}) = \{\zeta\}$ for each $\zeta \leq \omega_2 \cdot \omega_2$.

Then, we have $\mathrm{n}(\omega_2 \cdot \omega_2) = 1$, $\mathrm{n}(\omega_2 \cdot \omega_1) = 2$, $\mathrm{n}(\omega_2 \cdot \omega_1 + \omega) = 3$. Also, we have $j(\omega_2 \cdot \omega_2) = \mathrm{j}(\omega_2 \cdot \omega_1) = 1$ and $J(\omega_2 \cdot \omega_2) = J(\omega_2 \cdot \omega_1) = [0, \omega_2 \cdot \omega_2)$.

If $\mathrm{cf}(J(\zeta)^+) \in \{\kappa, \kappa^+\}$, we denote by $\{\epsilon^\zeta_\nu : \nu < \mathrm{cf}(J(\zeta)^+)\}$ the increasing enumeration of $\mathrm{E}(J(\zeta))$, i.e. $\epsilon^\zeta_\nu = \varepsilon^{J(\zeta)}_\nu$ for $\nu < \mathrm{cf}(J(\zeta)^+)$.

Now if $\zeta < \delta$, we define the *basic orbit* of ζ (with respect to $\mathbb{I}$) as

$$\mathrm{o}(\zeta) = \bigcup \{(\mathrm{E}(\mathrm{I}(\zeta, m)) \cap \zeta) : m < \mathrm{n}(\zeta)\}. \tag{10}$$

Note that this is the notion of orbit used in [5] in order to construct by forcing an LCS space X such that $\mathrm{SEQ}(X) = \langle \kappa \rangle_\eta$ for any specific regular cardinal κ and any ordinal $\eta < \kappa^{++}$. However, this notion of orbit can not be used to construct an LCS space X such that $\mathrm{SEQ}(X) = \langle \kappa \rangle_{\kappa^+} {}^\frown \langle \kappa^+ \rangle$. To check this point, assume on the contrary that such a space X can be constructed by forcing from the notion of a basic orbit. Then, since the basic orbit of κ^+ is $\{0\}$, we have that if x, y are any two different elements of $I_{\kappa^+}(X)$ and U, V are basic neighbourhoods of x, y respectively, then $U \cap V \subset I_0(X)$. But then, we deduce that $|I_1(X)| = \kappa^+$.

However, we will show that a refinement of the notion of basic orbit can be used to proof Theorem 1.6.

If $\zeta < \delta$ with $\mathrm{cf}\,\zeta \geq \kappa$, we define the *extended orbit* of ζ by

$$\overline{\mathrm{o}}(\zeta) = \mathrm{o}(\zeta) \cup (\mathrm{E}(J(\zeta)) \cap \zeta). \tag{11}$$

Consider the tree of intervals defined in Example-2.2. Then, we have $\mathrm{o}(\omega_2 \cdot \omega_1) = \overline{o}(\omega_2 \cdot \omega_1) = \{\omega_2 \cdot \xi : 0 \leq \xi < \omega_1\}$, $o(\omega_2 \cdot \omega_2) = \{0\}$, $\overline{o}(\omega_2 \cdot \omega_2) = \{\omega_2 \cdot \xi : 0 \leq \xi < \omega_2\}$.

Note that if $\zeta < \delta$, the basic orbit of ζ is a set of cardinality at most κ (see [5, Proposition 1.3]). Then, it is easy to see that for any $\zeta < \delta$ with $\operatorname{cf}\zeta \geq \kappa$, the extended orbit of ζ is a cofinal subset of ζ of cardinality $\operatorname{cf}\zeta$.

In order to define the desired notion of forcing, we need some preparations. The underlying set of the desired space will be the union of a collection of blocks.

Let

$$\mathbb{B} = \{S\} \cup \{\langle \zeta, \eta\rangle : \zeta < \delta, \operatorname{cf}\zeta \in \{\kappa, \kappa^+\}, \eta < \kappa^+\}. \tag{12}$$

Let

$$B_S = \delta \times \kappa \tag{13}$$

and

$$B_{\zeta,\eta} = \{\langle \zeta, \eta\rangle\} \times [\zeta, \delta) \times \kappa \tag{14}$$

for $\langle \zeta, \eta\rangle \in \mathbb{B} \setminus \{S\}$.

Let

$$X = \bigcup\{B_T : T \in \mathbb{B}\}. \tag{15}$$

The underlying set of our space will be X. We should produce a partition $X = Y \cup^* \bigcup^* \{Y_\zeta : \zeta \in \mathcal{L}^\delta_\kappa\}$ such that

(1) Y is an open subspace of X with $\mathrm{SEQ}(Y) = \langle \kappa\rangle_\delta$,

(2) $Y \cup Y_\zeta$ is an open subspace of X with $\mathrm{SEQ}(Y \cup Y_\zeta) = \langle \kappa\rangle_\zeta {}^\frown \langle \kappa^+\rangle_{\delta-\zeta}$.

We will have $Y = B_S$, $Y_\zeta = \bigcup\{B_{\zeta,\eta} : \eta < \kappa^+\}$ for $\zeta \in \mathcal{L}^\delta_\kappa$.

Let

$$\pi : X \longrightarrow \delta \text{ such that } \begin{array}{l} \pi(\langle \alpha, \nu\rangle) = \alpha, \\ \pi(\langle \zeta, \eta, \alpha, \nu\rangle) = \alpha. \end{array} \tag{16}$$

Let

$$\pi_- : X \longrightarrow \delta \text{ such that } \begin{array}{l} \pi_-(\langle \alpha, \nu\rangle) = \alpha, \\ \pi_-(\langle \zeta, \eta, \alpha, \nu\rangle) = \zeta. \end{array} \tag{17}$$

Define

$$\pi_B : X \longrightarrow \mathbb{B} \text{ by the formula } x \in B_{\pi_B(x)}. \tag{18}$$

Define the *block orbit* function $\mathrm{o_B} : \mathbb{B} \setminus \{S\} \longrightarrow \big[\delta\big]^{\leq\kappa}$ as follows:

$$\mathrm{o_B}(\langle \zeta, \eta\rangle) = \begin{cases} \overline{\mathrm{o}}(\zeta) & \text{if } \operatorname{cf}\zeta = \kappa, \\ \mathrm{o}(\zeta) \cup \{\epsilon^\zeta_\nu : \nu < \eta\} & \text{if } \operatorname{cf}\zeta = \kappa^+. \end{cases} \tag{19}$$

That is, if $\operatorname{cf}\zeta = \kappa^+$ then

$$\mathrm{o_B}(\langle\zeta,\eta\rangle) = \overline{\mathrm{o}}(\zeta) \cap \epsilon^{\zeta}_{\eta}.$$

Finally we define the *orbits* of the elements of X as follows:

(20)

$$\mathrm{o}^* : X \longrightarrow \big[\delta\big]^{\leq\kappa} \text{ such that } \begin{array}{l} \mathrm{o}^*(\langle\alpha,\nu\rangle) = \mathrm{o}(\alpha), \\ \mathrm{o}^*(\langle\zeta,\eta,\alpha,\nu\rangle) = \mathrm{o_B}(\langle\zeta,\eta\rangle) \cup (\mathrm{o}(\alpha)\setminus\zeta). \end{array}$$

Let $\Lambda \in \mathbb{I}$ and $\{x,y\} \in \big[X\big]^2$. We say that Λ *isolates x from y* if

(i) $\Lambda^- < \pi(x) < \Lambda^+$,
(ii) $\Lambda^+ \leq \pi(y)$ provided $\pi_B(x) = \pi_B(y)$,
(iii) $\Lambda^+ \leq \pi_-(y)$ provided $\pi_B(x) \neq \pi_B(y)$.

Now, we define the poset $\mathcal{P} = \langle P, \leq\rangle$ as follows: $\langle A, \preceq, i\rangle \in P$ iff

(P1) $A \in \big[X\big]^{<\kappa}$.
(P2) $\preceq$ is a partial order on A such that $x \preceq y$ implies $x = y$ or $\pi(x) < \pi(y)$.
(P3) Let $x \preceq y$.
  (a) If $\pi_B(y) = \langle\zeta,\eta\rangle$ and $\zeta \leq \pi(x)$ then $\pi_B(x) = \pi_B(y)$.
  (b) If $\pi_B(y) = \langle\zeta,\eta\rangle$ and $\zeta > \pi(x)$ then $\pi_B(x) = S$.
  (c) If $\pi_B(y) = S$ then $\pi_B(x) = S$.
(P4) $\mathrm{i} : \big[A\big]^2 \longrightarrow A \cup \{\text{undef}\}$ such that for each $\{x,y\} \in \big[A\big]^2$ we have

$$\forall a \in A([a \preceq x \wedge a \preceq y] \text{ iff } a \preceq \mathrm{i}\{x,y\}).$$

(P5) $\forall\{x,y\} \in \big[A\big]^2$ if x and y are $\preceq$-incomparable but $\preceq$-compatible, then $\pi(\mathrm{i}\{x,y\}) \in \mathrm{o}^*(x) \cap \mathrm{o}^*(y)$.
(P6) Let $\{x,y\} \in [A]^2$ with $x \preceq y$. Then:
  (a) If $\pi_B(x) = S$ and $\Lambda \in \mathbb{I}$ isolates x from y, then there is $z \in A$ such that $x \preceq z \preceq y$ and $\pi(z) = \Lambda^+$.
  (b) If $\pi_B(x) \neq S$, $\pi(x) \neq \pi_-(x)$ and $\Lambda \in \mathbb{I}$ isolates x from y, then there is $z \in A$ such that $x \preceq z \preceq y$ and $\pi(z) = \Lambda^+$.

The ordering on P is the extension: $\langle A, \preceq, \mathrm{i}\rangle \leq \langle A', \preceq', \mathrm{i}'\rangle$ iff $A' \subset A$, $\preceq' = \preceq \cap (A' \times A')$, and $\mathrm{i}' \subset \mathrm{i}$.

By using (P3), we obtain:

Claim 2.3. *Assume that x, y, z and Λ are as in (P6). Then we have:*
(a) If $\pi_B(x) = \pi_B(y)$, then $\pi_B(z) = \pi_B(x) = \pi_B(y)$.
(b) If $\pi_B(x) \neq \pi_B(y)$ and $\Lambda^+ < \pi_-(y)$, then $\pi_B(z) = \pi_B(x)$.
(c) If $\pi_B(x) \neq \pi_B(y)$ and $\Lambda^+ = \pi_-(y)$, then $\pi_B(z) = \pi_B(y)$.

Since $\kappa^{<\kappa} = \kappa$ implies $(\kappa^+)^{<\kappa} = \kappa^+$, we have that the cardinality of P is κ^+. Then, using the arguments of [5] it is enough to prove that Lemmas 2.4, 2.5 and 2.6 below hold.

Lemma 2.4. *$\mathcal{P}$ is κ-complete.*

Lemma 2.5. *$\mathcal{P}$ satisfies the κ^+-c.c.*

Lemma 2.6. *Assume that $p = \langle A, \preceq, \mathrm{i}\rangle \in P$, $x \in A$, and $\alpha < \pi(x)$. Then there is $p' = \langle A', \preceq', \mathrm{i}'\rangle \in P$ with $p' \leq p$ and there is $b \in A' \setminus A$ with $\pi(b) = \alpha$ such that $b \preceq' y$ iff $x \preceq y$ for $y \in A$.*

Since κ is regular, Lemma 2.4 clearly holds.

PROOF of Lemma 2.6. Let $\beta = \pi(x)$. Let K be a countable subset of $[\alpha, \beta)$ such that $\alpha \in K$ and $\mathrm{I}(\gamma, n)^+ \in K \cup [\beta, \delta)$ for $\gamma \in K$ and $n < \omega$. For each $\gamma \in K$ pick $b_\gamma \in X \setminus A$ such that $\pi(b_\gamma) = \gamma$ and

(1) if $\pi_B(x) = S$ then $\pi_B(b_\gamma) = S$.
(2) if $\pi_B(x) \neq S$ and $\gamma \geq \pi_-(x)$ then $\pi_B(b_\gamma) = \pi_B(x)$.
(3) if $\pi_B(x) \neq S$ and $\gamma < \pi_-(x)$ then $\pi_B(b_\gamma) = S$.

Let $A' = A \cup \{b_\gamma : \gamma \in K\}$,

$$\preceq' = \preceq \cup \{\langle b_\gamma, b_{\gamma'}\rangle : \gamma, \gamma' \in K, \gamma \leq \gamma'\} \\ \cup \{\langle b_\gamma, z\rangle : \gamma \in K, z \in A, x \preceq z\}.$$

The definition of i' is straightforward because if $y \in A'$ and $\gamma \in K$ then either y and b_γ are $\preceq'$-comparable or they are $\preceq'$-incompatible.

Then $p' = \langle A', \preceq', i'\rangle$ and $b = b_\alpha$ satisfy the requirements. □

Finally we should prove Lemma 2.5.

Proof of Lemma 2.5. Assume that $\langle r_\nu : \nu < \kappa^+\rangle \subset P$ with $r_\nu \neq r_\mu$ for $\nu < \mu < \kappa^+$.

Write $r_\nu = \langle A_\nu, \preceq_\nu, \mathrm{i}_\nu\rangle$ and $A_\nu = \{x_{\nu,i} : i < \sigma_\nu\}$.

Since we are assuming that $\kappa^{<\kappa} = \kappa$, by thinning out $\langle r_\nu : \nu < \kappa^+\rangle$ by means of standard combinatorial arguments, we can assume the following:

(A) $\sigma_\nu = \sigma$ for each $\nu < \kappa^+$.
(B) $\{A_\nu : \nu < \kappa^+\}$ forms a Δ-system with kernel A.
(C) For each $\nu < \mu < \kappa^+$ there is an isomorphism $h = h_{\nu,\mu} : \langle A_\nu, \preceq_\nu, \mathrm{i}_\nu\rangle \longrightarrow \langle A_\mu, \preceq_\mu, \mathrm{i}_\mu\rangle$ such that for every $i < \sigma$ and $x, y \in A_\nu$ the following holds:
  (a) $h \restriction A = \mathrm{id}$.
  (b) $h(x_{\nu,i}) = x_{\mu,i}$.
  (c) $\pi_B(x) = \pi_B(y)$ iff $\pi_B(h(x)) = \pi_B(h(y))$.
  (d) $\pi_B(x) = S$ iff $\pi_B(h(x)) = S$.
  (e) $\pi(x) = \pi_-(x)$ iff $\pi(h(x)) = \pi_-(h(x))$.
  (f) if $\{x, y\} \in [A]^2$ then $\mathrm{i}_\nu\{x, y\} = \mathrm{i}_\mu\{x, y\}$.

Note that in order to obtain (C)(f) we use condition (P5) and the fact that $|o^*(x)| \le \kappa$ for every $x \in A$. Also, we may assume the following:

(D) There is a partition $\sigma = K \cup^* F \cup^* L \cup^* D \cup^* M$ such that for each $\nu < \mu < \kappa^+$:
  (a) $\forall i \in K\ x_{\nu,i} \in A$ and so $x_{\nu,i} = x_{\mu,i}$. $A = \{x_{\nu,i} : i \in K\}$.
  (b) $\forall i \in F\ x_{\nu,i} \neq x_{\mu,i}$ but $\pi_B(x_{\nu,i}) = \pi_B(x_{\mu,i}) \neq S$.
  (c) $\forall i \in L\ \pi_B(x_{\nu,i}) \neq \pi_B(x_{\mu,i})$ but $\pi_-(x_{\nu,i}) = \pi_-(x_{\mu,i})$.
  (d) $\forall i \in D\ \pi_B(x_{\nu,i}) = S$ and $\pi(x_{\nu,i}) \neq \pi(x_{\mu,i})$.
  (e) $\forall i \in M\ \pi_B(x_{\nu,i}) \neq S$ and $\pi_-(x_{\nu,i}) \neq \pi_-(x_{\mu,i})$.

(E) If $\pi_B(x_{\nu,i}) = \pi_B(x_{\nu,j})$ then $\{i,j\} \in \big[K \cup D\big]^2 \cup \big[K \cup F\big]^2 \cup \big[L\big]^2 \cup \big[M\big]^2$.

It is well-known that if $\gamma < \kappa = \kappa^{<\kappa}$ then the following partition relation holds:

$$\kappa^+ \longrightarrow (\kappa^+, (\omega)_\gamma)^2.$$

Hence we can assume:

(F) If $\nu < \mu < \kappa^+$ then for each $i \in \sigma$ we have
  (a) $\pi(x_{\nu,i}) \le \pi(x_{\mu,i})$,
  (b) $\pi_-(x_{\nu,i}) \le \pi_-(x_{\mu,i})$.

By (F)(a) and (F)(b) the sequences $\{\pi(x_{\nu,i}) : \nu < \kappa^+\}$ and $\{\pi_-(x_{\nu,i}) : \nu < \kappa^+\}$ are increasing for each $i \in \sigma$, hence the following definition is meaningful:

For $i \in \sigma$ let

$$\delta_i = \begin{cases} \pi(x_{\nu,i}) & \text{if } i \in K, \\ \sup\{\pi(x_{\nu,i}) : \nu < \kappa^+\} & \text{if } i \in F \cup D, \\ \pi_-(x_{\nu,i}) & \text{if } i \in L, \\ \sup\{\pi_-(x_{\nu,i}) : \nu < \kappa^+\} & \text{if } i \in M. \end{cases}$$

By using Proposition 2.1, (C)(c) and condition (P3), we obtain:

Claim 2.7. *(a) If $i \in F \cup D \cup M$, then $cf(\delta_i) = \kappa^+$ and $sup(J(\delta_i)) = \delta_i$. Moreover for every $\nu < \kappa^+$ we have $\pi(x_{\nu,i}) < \delta_i$ if $i \in F \cup D$, and $\pi_-(x_{\nu,i}) < \delta_i$ if $i \in M$.*
(b) If $\{i,j\} \in [L]^2 \cup [M]^2$ and $x_{\nu,i} \prec_\nu x_{\nu,j}$ for $\nu < \kappa^+$, then $\delta_i = \delta_j$.

Indeed, (b) holds for large enough ν, and so (C)(c) implies that it holds for each ν.

We put

$$Z_0 = \{\pi_-(x_{\nu,i}) : i \in F \cup K, \pi_B(x_{\nu,i}) \neq S\} \cup \{\delta_i : i \in \sigma\}. \tag{21}$$

Since $\pi''A = \{\delta_i : i \in K\}$ we have $\pi''A \subset Z_0$. Then, we define Z as the closure of Z_0 with respect to $\mathbb{I}$:

$$Z = Z_0 \cup \{I^+ : I \in \mathbb{I}, I \cap Z_0 \neq \emptyset\}. \tag{22}$$

Since $|Z| < \kappa$, we can assume:

(G) $A = \{x_{\nu,i} : i \in K \cup F \cup D, \pi(x_{\nu,i}) \in Z\}$.

Equivalently,

$$\text{if } i \in F \cup D \text{ then } \pi(x_{\nu,i}) \notin Z. \tag{23}$$

Let us remark that for $i \in L \cup M$ we may have that $\pi(x_{\nu,i}) \in Z$.

Our aim is to show that there are $\nu < \mu < \kappa^+$ such that r_ν and r_μ are compatible. Note that if $x, y \in A$ with $x \neq y$ then, by (C)(f), we may assure that $i_\nu\{x, y\} = i_\mu\{x, y\}$. However, if $x \in A_\nu \setminus A$ and $y \in A_\mu \setminus A$ it may happen that for infinitely many $v \in A$ we have $v \preceq_\nu x$ and $v \preceq_\mu y$. Then, in order to amalgamate r_ν and r_μ in such a way that any pair of such elements has an infimum in the amalgamation, we will need to add new elements to $A_\nu \cup A_\mu$. Then, the next definitions will permit us to find suitable room for adding new elements to the domains of the conditions.

Let

$$\sigma_1 = \{i \in \sigma \setminus K : \mathrm{cf}(\delta_i) = \kappa\}$$

and

$$\sigma_2 = \{i \in \sigma \setminus K : \mathrm{cf}(\delta_i) = \kappa^+\}.$$

Assume that $i \in \sigma \setminus K$. Put $I_i = J(\delta_i)$. Let

$$\xi_i = \min\{\nu \in \mathrm{cf}\,\delta_i : \epsilon^{I_i}_\nu > \sup(\delta_i \cap Z)\}.$$

Then, if $i \in \sigma_1$ we put

$$\underline{\gamma}(\delta_i) = \epsilon^{I_i}_{\xi_i} \text{ and } \gamma(\delta_i) = \delta_i,$$

and if $i \in \sigma_2$ we put

$$\underline{\gamma}(\delta_i) = \epsilon^{I_i}_{\xi_i} \text{ and } \gamma(\delta_i) = \epsilon^{I_i}_{\xi_i+\kappa}.$$

Claim 2.8. *For each $i \in F \cup D \cup M$ there is $\nu_i < \kappa^+$ such that for all $\nu_i \leq \nu < \kappa^+$ we have:*

$$\textit{if } i \in F \cup D \textit{ then } \pi(x_{\nu,i}) \in J(\delta_i) \setminus \gamma(\delta_i) \tag{24}$$

and

$$\text{if } i \in M \text{ then } \pi_-(x_{\nu,i}) \in J(\delta_i) \setminus \gamma(\delta_i). \tag{25}$$

Proof. For $i \in F \cup D \cup M$ we have

$$\delta_i = \begin{cases} \sup\{\pi(x_{\nu,i}) : \nu < \kappa^+\} & \text{if } i \in F \cup D, \\ \sup\{\pi_-(x_{\nu,i}) : \nu < \kappa^+\} & \text{if } i \in M, \end{cases} \tag{26}$$

and $\gamma(\delta_i) < \sup(\mathrm{J}(\delta_i)) = \delta_i$. □

Claim 2.9. *For each $i \in L$ with $cf(\delta_i) = \kappa^+$ there is $\nu_i < \kappa^+$ such that for all $\nu_i \le \nu < \kappa^+$, $o^*(x_{\nu,i}) \supset \overline{o}(\delta_i) \cap \gamma(\delta_i)$.*

Definition 2.10. r_ν is *good* iff

(i) $\forall i \in F \cup D$ $\pi(x_{\nu,i}) \in \mathrm{J}(\delta_i) \setminus \gamma(\delta_i)$.
(ii) $\forall i \in M$ $\pi_-(x_{\nu,i}) \in J(\delta_i) \setminus \gamma(\delta_i)$.
(iii) $\forall i \in L$ if $\mathrm{cf}\, \delta_i = \kappa^+$ then $\mathrm{o}^*(x_{\nu,i}) \supset \overline{o}(\delta_i) \cap \gamma(\delta_i)$.

Using Claims 2.8 and 2.9 we can assume:
(H) r_ν is good for $\nu < \kappa^+$.

By using (H), we will prove that r_ν and r_μ are compatible for $\{\nu, \mu\} \in [\kappa^+]^2$. First, we need to prove some fundamental facts.
By using (P3), (E) and (C)(c) we obtain:

Claim 2.11. *If $x_{\nu,i} \preceq_\nu x_{\nu,j}$ then either $\pi_B(x_{\nu,i}) = S$ or $\pi_B(x_{\nu,i}) = \pi_B(x_{\nu,j})$ and $\{i,j\} \in [K \cup F]^2 \cup [L]^2 \cup [M]^2$.*

Indeed, (P3) and (E) imply that Claim 2.11 holds for large enough ν, and then (C)(c) yields that it holds for each ν.

Claim 2.12. *If $x_{\nu,i} \preceq_\nu x_{\nu,j}$ then $\delta_i \le \delta_j$.*

Proof. If $x_{\nu,i} \preceq_\nu x_{\nu,j}$ then $x_{\mu,i} \preceq_\mu x_{\mu,j}$ for each $\mu < \kappa^+$, and so we have:
(a) $\pi(x_{\mu,i}) \le \pi(x_{\mu,j})$,
(b) $\pi_-(x_{\mu,i}) \le \pi_-(x_{\mu,j})$,
(c) if $\pi_B(x_{\mu,i}) \ne \pi_B(x_{\mu,j})$ then $\pi(x_{\mu,i}) \le \pi_-(x_{\mu,j})$.
Hence if $\pi_B(x_{\nu,i}) \ne \pi_B(x_{\nu,j})$ then

$$\delta_i = \sup\{\pi(x_{\mu,i}) : \mu < \kappa^+\} \le \sup\{\pi_-(x_{\mu,j}) : \mu < \kappa^+\} \le \delta_j. \tag{27}$$

If $\pi_B(x_{\nu,i}) = \pi_B(x_{\nu,j})$ then either $\{i,j\} \in \left[K \cup F\right]^2 \cup \left[K \cup D\right]^2$ and so

$$\delta_i = \sup\{\pi(x_{\mu,i}) : \mu < \kappa^+\} \le \sup\{\pi(x_{\mu,j}) : \mu < \kappa^+\} = \delta_j, \tag{28}$$

or $\{i,j\} \in \left[L\right]^2 \cup \left[M\right]^2$ and so

$$\delta_i = \sup\{\pi_-(x_{\mu,i}) : \mu < \kappa^+\} \le \sup\{\pi_-(x_{\mu,j}) : \mu < \kappa^+\} = \delta_j. \tag{29}$$

□

Claim 2.13. *Assume $i, j \in \sigma$. If $x_{\nu,i} \preceq_\nu x_{\nu,j}$ then either $\delta_i = \delta_j$ or there is $a \in A$ with $x_{\nu,i} \preceq_\nu a \preceq_\nu x_{\nu,j}$.*

Proof. Put $x_i = x_{\nu,i}, x_j = x_{\nu,j}$. Assume that $i, j \notin K$ and $\delta_i \neq \delta_j$. By Claim 2.12, we have $\delta_i < \delta_j$. Since $i \in L \cup M$ implies $\delta_i = \delta_j$, we have that $i \in F \cup D$, and so $\pi(x_i) < \delta_i$, $\operatorname{cf}(\delta_i) = \kappa^+$ and $J(\delta_i)^+ = \delta_i$. We distinguish the following cases:

Case 1. $i \in D$ and $j \in D \cup L \cup M$.

Since $\delta_i < \delta_j$, we have that $J(\delta_i)$ isolates x_i from x_j. Also, note that if $j \in L \cup M$, then $J(\delta_i)^+ = \delta_i < \pi_-(x_j)$. By (P6)(a), we infer that there is an $x = x_{\nu,k} \in A_\nu$ such that $\pi(x) = \delta_i$ and $x_i \prec_\nu x \prec_\nu x_j$. Now, by Claim 2.3(a)-(b), we deduce that $k \in K \cup D$. But as $\delta_i \in Z$, by (G), we have that $x \in A$, and so we are done.

Case 2. $i \in D$ and $j \in F$.

We have that $\pi_B(x_i) \neq \pi_B(x_j)$. By using (P3), we infer that $\delta_i \leq \pi_-(x_j)$, and so $J(\delta_i)$ isolates x_i from x_j. If $\delta_i < \pi_-(x_j)$, we proceed as in Case 1. So, assume that $\delta_i = \pi_-(x_j)$. By (P6)(a), we deduce that there is an $x = x_{\nu,k} \in A_\nu$ such that $\pi(x) = \delta_i$ and $x_i \prec_\nu x \prec_\nu x_j$. By Claim 2.3(c), we infer that $k \in K \cup F$. Then as $\delta_i \in Z$, we have that $x \in A$ by (G).

Case 3. $i, j \in F$.

We have that $\pi_B(x_i) = \pi_B(x_j) \neq S$ and $J(\delta_i)$ isolates x_i from x_j. Since $\pi_-(x_i) \in Z$ and we are assuming that $i \notin K$, we infer that $\pi(x_i) \neq \pi_-(x_i)$. Now, applying (P6)(b), we deduce that there is an $x = x_{\nu,k} \in A_\nu$ such that $\pi(x) = \delta_i$ and $x_i \prec_\nu x \prec_\nu x_j$. Now we deduce from Claim 2.3(a) that $k \in K \cup F$. Then as $\delta_i \in Z$, we have that $x \in A$ by (G). $\square$

Claim 2.14. *If $x \in A$ and $y \in A_\nu$, and x and y are compatible but incomparable in r_ν, then $\mathrm{i}_\nu\{x, y\} \in A$.*

Proof. Indeed, $\pi(\mathrm{i}_\nu\{x, y\}) \in \mathrm{o}^*(x)$ by (P5) and $|\mathrm{o}^*(x)| \leq \kappa$. $\square$

Claim 2.15. *Assume that $x_{\nu,i}$ and $x_{\nu,j}$ are compatible but incomparable in r_ν. Let $x_{\nu,k} = \mathrm{i}_\nu\{x_{\nu,i}, x_{\nu,j}\}$. Then either $x_{\nu,k} \in A$ or $\delta_i = \delta_j = \delta_k$.*

Proof. Assume $x_{\nu,k} \notin A$. Then $k \notin K$. If $\delta_k \neq \delta_i$, we infer that there is $b \in A$ with $x_{\nu,k} \preceq_\nu b \preceq_\nu x_{\nu,i}$ by Claim 2.13.. So $x_{\nu,k} = \mathrm{i}_\nu\{b, x_{\nu,j}\}$ and thus $x_{\nu,k} \in A$ by Claim 2.14, contradiction.

Thus $\delta_i = \delta_k$, and similarly $\delta_j = \delta_k$. $\square$

After this preparation fix $\{\nu, \mu\} \in \left[\kappa^+\right]^2$. We do not assume that $\nu < \mu$! Let $p = r_\nu$ and $q = r_\mu$. Our purpose is to show that p and q

are compatible. Write $p = \langle A_p, \preceq_p, \mathrm{i}_p \rangle$ and $q = \langle A_q, \preceq_q, \mathrm{i}_q \rangle$, $x^p_i = x_{\nu,i}$ and $x^q_i = x_{\mu,i}$, $\delta_{x^p_i} = \delta_{x^q_i} = \delta_i$.

If $s = x^p_i$ write $s \in K$ iff $i \in K$. Define $s \in L$, $s \in F$, $s \in M$, $s \in D$ similarly.

In order to amalgamate conditions p and q, we will use a refinement of the notion of amalgamation given in [5, Definition 2.4].

Let $A' = \{x^p_i : i \in F \cup D \cup M \cup L\}$.

Let $\mathrm{rk} : \langle A', \preceq_p \restriction A' \rangle \longrightarrow \theta$ be an order-preserving injective function for some ordinal $\theta < \kappa$.

For $x \in A'$, by induction on $\mathrm{rk}(x) < \theta$ choose $\beta_x \in \delta$ as follows:

Assume that $\mathrm{rk}(x) = \tau$ and β_z is defined provided $\mathrm{rk}(z) < \tau$.

Let

$$\beta_x = \min\big((\overline{\mathrm{o}}(\delta_x) \cap [\underline{\gamma}(\delta_x), \gamma(\delta_x))) \setminus \sup\{\beta_z : z \prec_p x\} \big). \tag{30}$$

Since $z \preceq_p x$ implies $\delta_z \le \delta_x$ by Claim 2.12, we have $\beta_z < \gamma(\delta_x)$ for $z \prec_p x$. Since $\mathrm{cf}(\gamma(\delta_x)) = \kappa$ and $|A'| < \kappa$ we have $\sup\{\beta_z : z \prec_p x\} < \gamma(\delta_x)$, so β_x is always defined.

For $x \in A'$ let

$$y_x = \begin{cases} \langle \beta_x, \mathrm{rk}(x) \rangle & \text{if } x \in L \cup D \cup M, \\ \langle \zeta, \eta, \beta_x, \mathrm{rk}(x) \rangle & \text{if } x \in F,\ \pi_B(x) = \langle \zeta, \eta \rangle. \end{cases} \tag{31}$$

Put

$$Y = \{y_x : x \in A'\}. \tag{32}$$

For $x \in A'$ put

$$g(y_x) = x \text{ and } \bar{g}(y_x) = x', \tag{33}$$

where x' is the "twin" of x in A_q (i.e. $h_{\nu,\mu}(x) = x'$).

We will include the elements of Y in the domain of the amalgamation r of p and q. In this way, we will be able to define the infimum in r of elements s, t where $s \in A_p \setminus A_q$ and $t \in A_q \setminus A_p$.

We need to prove some basic facts.

Claim 2.16. *If $x \in A'$ then*

$$\overline{\mathrm{o}}(\delta_x) \cap [\underline{\gamma}(\delta_x), \gamma(\delta_x)) \subset o^*(x) \cap o^*(x').$$

Proof. Let $\alpha \in \overline{\mathrm{o}}(\delta_x) \cap [\underline{\gamma}(\delta_x), \gamma(\delta_x))$. It is enough to show that $\alpha \in o^*(x)$. Note that if $x \in D$, then $\alpha \in o(\pi(x)) = o^*(x)$. If $x \in M$, we have that $\alpha \in o(\pi_-(x)) \subset o_B(\pi_B(x)) \subset o^*(x)$. Also, if $x \in L$ then as p is good we have that $\alpha \in o_B(\pi_B(x)) \subset o^*(x)$. Now, assume that $x \in F$. Since $\pi_-(x) \in Z$, we have that $\pi_-(x) < \underline{\gamma}(\delta_x)$, hence $\alpha \in o(\pi(x)) \setminus \pi_-(x)$, and so $\alpha \in o^*(x)$. $\square$

Note that we obtain as an immediate consequence of Claim 2.16 that $\beta_x \in o^*(x) \cap o^*(x')$ for every $x \in A'$.

Claim 2.17. *If $x \in A'$ then*

$$\mathrm{o}^*(y_x) \supset (\mathrm{o}^*(x) \cap \pi(y_x)) \cup \{\beta_z : \delta_z = \delta_x \wedge z \prec_p x\}. \tag{34}$$

Proof. Note that if $I \in \mathbb{I}$ and $\alpha, \beta \in E(I)$ with $\alpha < \beta$, we have that $\alpha \in o(\beta)$. By using this fact, it is easy to verify that $\{\beta_z : \delta_z = \delta_x$ and $z \prec_p x\} \subset o^*(y_x)$.

Now we prove that $o^*(y_x) \supset o^*(x) \cap \pi(y_x)$. Suppose that $\zeta \in o^*(x) \cap \pi(y_x)$. We distinguish the following three cases:

Case 1. $x \in D$.

Then $x, y_x \in B_S$, and so we have $o^*(x) = o(\pi(x))$ and $o^*(y_x) = o(\pi(y_x)) = o(\beta_x)$. Let $k = j(\delta_x)$, i.e. $J(\delta_x) \in \mathcal{I}_k$. Since $\zeta \in o(\pi(x)) \cap \pi(y_x)$, we infer that $\zeta \in E(I(\pi(x), m)) \cap \pi(y_x)$ for some $m \leq k$. Note that for $m \leq k$ we have $I(\pi(x), m) = I(\pi(y_x), m)$. So, $\zeta \in o(\pi(y_x)) = o^*(y_x)$.

Case 2. $x \in L \cup M$.

Since $\zeta \in o^*(x) \cap \pi(y_x)$, we infer that $\zeta \in o_B(\pi_B(x))$. Then as $y_x \in B_S$, we can show that $\zeta \in o(\pi(y_x)) = o^*(y_x)$ by using an argument similar to the one given in Case 1.

Case 3. $x \in F$.

We have $\pi_B(x) = \pi_B(y_x) \neq S$. Put $(\xi, \eta) = \pi_B(x) = \pi_B(y_x)$. So,
$o^*(x) = o_B((\xi, \eta)) \cup (o(\pi(x)) \setminus \pi_-(x))$,
$o^*(y_x) = o_B((\xi, \eta)) \cup (o(\pi(y_x)) \setminus \pi_-(x))$.

So we may assume that $\zeta \in o(\pi(x)) \setminus \pi_-(x)$, and then we can proceed as in Case 1. □

Claim 2.18. *There are no $y \in Y$ and $a \in A$ such that $a \preceq_p g(y), \overline{g}(y)$ and $\pi(y) \leq \pi(a)$.*

Proof. Assume that $y \in Y$. Put $x = g(y)$ and $I = J(\delta_x)$. Note that if $x \in F \cup D \cup M$, then since $\sup(I \cap Z) < \underline{\gamma}(\delta_x)$ we infer that there is no $a \in A$ such that $a \preceq_p x$ and $\pi(a) \geq \pi(y)$.

Now, suppose that $x \in L$. Note that there is no $a \in A$ such that $a \prec_p x$ and $\pi_B(a) = \pi_B(x)$. Also, as $\sup(\delta_x \cap Z) < \underline{\gamma}(\delta_x)$, we infer that there is no $a \in A \cap B_S$ such that $a \preceq_p x$ and $\pi(a) \geq \pi(y)$. □

Claim 2.19. *If $x \in F \cup D \cup M$, then there is no interval that isolates y_x from x.*

Proof. By Claim 2.7(a), we have $\mathrm{cf}(\delta_x) = \kappa^+$ and $\pi(x) < \delta_x$. By Proposition-2.1, we have $j(\delta_x) = n(\delta_x)$ and $\delta_x = J(\delta_x)^+$. Then, assume

on the contrary that there is an interval $\Lambda \in \mathbb{I}$ that isolates y_x from x. Let $m < \omega$ such that $\Lambda = I(\pi(y_x), m)$. As Λ isolates y_x from x and $x, y_x \in J(\delta_x)$, we deduce that $m > j(\delta_x)$. But from $m > j(\delta_x)$ and $\pi(y_x) \in E(J(\delta_x))$ we infer that $\pi(y_x) = \Lambda^-$. Hence, Λ does not isolate y_x from x. $\square$

However, if $x \in L$ it may happen that there is a $\Lambda \in \mathbb{I}$ that isolates y_x from x.

Now, we are ready to start to define the common extension $r = (A_r, \prec_r, i_r)$ of p and q. First, we define the universe A_r. Put $L^+ = \{x \in L : \pi(x) \neq \pi_-(x)\}$. Then, if $x \in L^+$ and x' is the twin element of x, we consider new elements $u_x, u_{x'} \in X \setminus (A_p \cup A_q \cup Y)$ such that $\pi_B(u_x) = \pi_B(x)$, $\pi(u_x) = \pi_-(x)$, $\pi_B(u_{x'}) = \pi_B(x')$ and $\pi(u_{x'}) = \pi_-(x')$. We suppose that $u_x, u_z, u_{x'}, u_{z'}$ are different if x, z are different elements of L^+. We put $U = \{u_x : x \in L^+\}$ and $U' = \{u_{x'} : x \in L^+\}$. Then, we define

$$A_r = A_p \cup A_q \cup Y \cup U \cup U'.$$

Clearly, A_r satisfies (P1). Now, our purpose is to define $\preceq_r$. First, for $x, y \in \left[A_p \cup A_q\right]^2$ let

(35) $x \preceq_{p,q} y$ iff $\exists z \in A_p \cup A_q\ [x \preceq_p z \vee x \preceq_q z] \ \wedge\ [z \preceq_p y \vee z \preceq_q y]$.

The following claim is straightforward.

Claim 2.20. *$\preceq_{p,q}$ is the partial order on $A_p \cup A_q$ generated by $\preceq_p \cup \preceq_q$.*

Next, we define the relation $\preceq^*$ on $A_p \cup A_q \cup Y$ as follows. Let us recall that $A = A_p \cap A_q$. Informally, $\preceq^*$ will be the ordering on $A_p \cup A_q \cup Y$ generated by

$$\begin{aligned} \preceq_{p,q} \cup \{\langle y, g(y)\rangle, \langle y, \bar{g}(y)\rangle : y \in Y\} \cup \\ \{\langle y, y'\rangle : y, y' \in Y, g(y) \preceq_p g(y')\} \cup \\ \{\langle a, y\rangle : a \in A, y \in Y, a \preceq_p g(y)\}. \end{aligned}$$

The formal definition is a bit different, but its formulation simplifies the separation of different cases later. So we introduce five relations on $A_p \cup A_q \cup Y$ as follows:

$$\begin{aligned} \prec^{R1_p} &= \{\langle y, a\rangle : y \in Y, a \in A_p, g(y) \preceq_p a\}, \\ \prec^{R1_q} &= \{\langle y, a\rangle : y \in Y, a \in A_q, \bar{g}(y) \preceq_q a\}, \\ \preceq^{R2} &= \{\langle y, y'\rangle : y, y' \in Y, g(y) \preceq_p g(y')\}, \\ \prec^{R3_p} &= \{\langle x, y\rangle : x \in A_p, y \in Y, \exists a \in A\ x \preceq_p a \preceq_p g(y)\}, \\ \prec^{R3_q} &= \{\langle x, y\rangle : x \in A_q, y \in Y, \exists a \in A\ x \preceq_q a \preceq_q \bar{g}(y)\}. \end{aligned}$$

Then, we put

$$\preceq^* = \preceq_{p,q} \cup \prec^{R1_p} \cup \prec^{R1_q} \cup \preceq^{R2} \cup \prec^{R3_p} \cup \prec^{R3_q} . \tag{36}$$

The partial order $\preceq_r$ will be an extension of $\preceq^*$. So, we need to prove the following lemma:

Lemma 2.21. $\preceq^*$ *is a partial order on* $A_p \cup A_q \cup Y$.

Proof. Let $s \preceq_r t \preceq_r u$. We should show that $s \preceq_r u$.

We can assume that $t \notin A_q \setminus A_p$.

Case I. $s \in A_p \cup A_q$, $t \in A_p$ *and* $s \preceq_{p,q} t$.

Without loss of generality, we may assume that $u \in Y$ and $t \prec^{R3p} u$, i.e. there is $a \in A$ such that $t \preceq_p a \preceq_p g(u)$.

Case I.1. $s \in A_p$.

Then $s \preceq_p a \preceq_p g(u)$ and so $s \prec^{R3p} u$.

Case I.2. $s \in A_q \setminus A_p$.

Then there is $b \in A$ such that $s \preceq_q b \preceq_p t \preceq_p a \preceq_p g(u)$. Then $s \preceq_q a \preceq_q \bar{g}(u)$ so $s \prec^{R3q} u$.

Case II. $s \in Y$, $t \in A_p$ *and* $s \prec^{R1p} t$.

Case II.1. $u \in A_p \cup A_q$ *and* $s \prec^{R1p} t \preceq_{p,q} u$.

Case II.1.i. $u \in A_p$.

Then $g(s) \preceq_p t \preceq_p u$ hence $s \prec^{R1p} u$.

Case II.1.ii. $u \in A_q \setminus A_p$.

Then there is $a \in A$ such that $g(s) \preceq_p t \preceq_p a \preceq_q u$. Hence $\bar{g}(s) \preceq_q a \preceq_q u$ and so $\bar{g}(s) \preceq_q u$. Thus $s \prec^{R1q} u$.

Case II.2. $u \in Y$ *and* $s \prec^{R1p} t \prec^{R3p} u$.

Then there is $a \in A$ such that $g(s) \preceq_p t \preceq_p a \preceq_p g(u)$ and so $s \preceq^{R2} u$.

Case III. $s, t \in Y$ *and* $s \preceq^{R2} t$.

Case III.1. $u \in A_p$ *and* $s \preceq^{R2} t \prec^{R1p} u$.

Then $g(s) \preceq_p g(t) \preceq_p u$ so $s \prec^{R1p} u$.

Case III.2. $u \in A_q$ *and* $s \preceq^{R2} t \prec^{R1q} u$.

Then $g(s) \preceq_p g(t)$ and $\bar{g}(t) \preceq_q u$. Thus $\bar{g}(s) \preceq_q \bar{g}(t) \preceq_q u$ so $s \prec^{R1q} u$.

Case III.3. *$u \in Y$ and $s \preceq^{R2} t \preceq^{R2} u$.*

Then $g(s) \preceq_p g(t) \preceq_p g(u)$ so $s \preceq^{R2} u$.

Case IV. *$s \in A_p$, $t \in Y$ and $s \prec^{R3p} t$.*

Case IV.1. *$u \in A_p$ and $s \prec^{R3p} t \prec^{R1p} u$.*

Then there is $a \in A$ such that $s \preceq_p a \preceq_p g(t) \preceq_p u$ so $s \preceq_p u$.

Case IV.2. *$u \in A_q$ and $s \prec^{R3p} t \prec^{R1q} u$.*

Then there is $a \in A$ such that $s \preceq_p a \preceq_p g(t)$ and $\bar{g}(t) \preceq_q u$. So $a \preceq_q \bar{g}(t)$ and hence $s \preceq_p a \preceq_q u$. Thus $s \preceq_{p,q} u$.

Case IV.3. *$u \in Y$ and $s \prec^{R3p} t \preceq^{R2} u$.*

Then there is $a \in A$ such that $s \preceq_p a \preceq_p g(t) \preceq_p g(u)$ and so $s \prec^{R3p} u$.

Case V. *$s \in A_q$, $t \in Y$ and $s \prec^{R3q} t$.*

Only case (3) is different from (IV):

Case V.3. *$u \in Y$ and $s \prec^{R3q} t \preceq^{R2} u$.*

Then there is $a \in A$ such that $s \preceq_q a \preceq_q \bar{g}(t)$ and $g(t) \preceq_p g(u)$. Then $\bar{g}(t) \preceq_q \bar{g}(u)$, so $s \preceq_q a \preceq_q \bar{g}(u)$, thus $s \prec^{R3q} u$. □

Informally, $\preceq_r$ will be the ordering on $A_p \cup A_q \cup Y \cup U \cup U'$ generated by

$$\preceq^* \cup \{\langle y_s, u_s\rangle : s \in A_p \cup A_q\} \cup \{\langle u_s, s\rangle : s \in A_p \cup A_q\}.$$

Now, in order to define $\preceq_r$ we need to make the following definitions:

$$\begin{aligned}
\prec^{R4_p} &= \{\langle s, u_x\rangle : s \in A_p \cup A_q \cup Y, x \in L^+ \text{ and } s \preceq^* y_x\},\\
\prec^{R4_q} &= \{\langle s, u_{x'}\rangle : s \in A_p \cup A_q \cup Y, x \in L^+ \text{ and } s \preceq^* y_x\},\\
\prec^{R5_p} &= \{\langle u_x, t\rangle : x \in L^+, t \in A_p \text{ and } x \preceq_p t\},\\
\prec^{R5_q} &= \{\langle u_{x'}, t\rangle : x \in L^+, t \in A_q \text{ and } x' \preceq_q t\},\\
=^{U} &= \{\langle u_x, u_x\rangle : x \in L^+\},\\
=^{U'} &= \{\langle u_{x'}, u_{x'}\rangle : x \in L^+\}.
\end{aligned}$$

Then, we define:

$$\preceq_r = \preceq^* \cup \prec^{R4_p} \cup \prec^{R4_q} \cup \prec^{R5_p} \cup \prec^{R5_q} \cup =^{U} \cup =^{U'} . \tag{37}$$

Write $x \prec_r y$ iff $x \preceq_r y$ and $x \neq y$.

Lemma 2.22. *$\preceq_r$ is a partial order on A_r.*

Proof. Assume that $s \prec_r t \prec_r v$. We have to show that $s \prec_r v$. Note that if $s, t, v \in A_p \cup A_q \cup Y$, then $s \prec^* t \prec^* v$, and so we are done by Lemma 2.21. Also, it is impossible that two elements of $\{s, t, v\}$ are in $U \cup U'$. To check this point, assume that $s, v \in U$. Put $s = u_x$, $v = u_z$ for $x, z \in L^+$. As $u_x \prec_r t$, we have $u_x \prec^{R5p} t$ and so $x \preceq_p t$. As $t \prec_r u_z$, we have $t \prec^{R4p} u_z$ and so $t \prec^* y_z$. Hence, $x \preceq_p t \prec^* y_z \prec^* z$. Since $x \preceq_p t$ and $x \in L$, we infer that $t \in L$. Also, from $t \prec^* y_z$ we deduce that $t \prec^{R3p} y_z$ and so there is an $a \in A$ such that $t \preceq_p a \preceq_p z$. But since $t \in L$, it is impossible that there is an $a \in A$ with $t \preceq_p a$. Proceeding in an analogous way, we arrive to a contradiction if we assume that $s \in U$ and $v \in U'$. So, at most one element of $\{s, t, v\}$ is in $U \cup U'$. Then, we consider the following cases:

Case 1. $s \in U$.

We have that $t, v \in A_p \cup A_q \cup Y$. Put $s = u_x$ for some $x \in L^+$. Since $u_x \prec_r t$, we have $u_x \prec^{R5p} t$ and so $x \preceq_p t$. As $t \prec_r v$, we have $t \prec^* v$. So, $x \preceq_p t \prec^* v$. But as $x \in L$ and $x \preceq_p t$, we infer that $t \in L$. Hence, $t \prec_p v$. Thus $x \prec_p v$, therefore $u_x \prec^{R5p} v$, and so $u_x \prec_r v$.

Case 2. $t \in U$.

We have that $s, v \in A_p \cup A_q \cup Y$. Put $t = u_x$ for $x \in L^+$. From $s \prec_r u_x$, we infer that $s \prec^{R4p} u_x$ and so $s \preceq^* y_x$. From $u_x \prec_r v$, we deduce that $u_x \prec^{R5p} v$ and hence $x \preceq_p v$. So we have $s \preceq^* y_x \prec^* x \preceq_p v$, and therefore $s \prec_r v$.

Case 3. $v \in U$.

We have that $s, t \in A_p \cup A_q \cup Y$. Put $v = u_x$ for $x \in L^+$. Since $t \prec_r u_x$, we have that $t \prec^{R4p} u_x$ and so $t \preceq^* y_x$. And from $s \prec_r t$ we deduce that $s \prec^* t$. So $s \prec^* y_x$, hence $s \prec^{R4p} u_x$, and thus $s \prec_r u_x$. $\square$

Now note that $s \prec^{R3_p} t$ implies $\pi(s) < \pi(t)$ by Claim 2.18, and so it is clear that $s \prec_r t$ implies $\pi(s) < \pi(t)$. Thus, condition (P2) holds. Also, it is easy to verify that $\preceq_r$ satisfies (P3).

If $x \in A_p$ denote its "twin" in A_q by x', and vice versa, if $x \in A_q$ denote its "twin" in A_p by x'.

Extend the definition of g as follows: $g : A_r \longrightarrow A_p$ is a function,

$$g(x) = \begin{cases} x & \text{if } x \in A_p, \\ x' & \text{if } x \in A_q, \\ s & \text{if } x = y_s \text{ for some } s \in A_p, \\ t & \text{if } x = u_t \text{ for some } t \in A_p, \\ t' & \text{if } x = u_t \text{ for some } t \in A_q. \end{cases}$$

For $\{s,t\} \in \left[A_r\right]^2$ we will be able to define the infimum of s,t in $(A_r, \preceq_r)$ from the infimum of $g(s), g(t)$ in p. Now, we need to prove some facts concerning the behavior of the function g on A_r.

Claim 2.23. *Let $a \in A$ and $x \in A_r$. Then*
(1) $x \preceq_r a$ iff $g(x) \preceq_p a$,
(2) $a \preceq_r x$ iff $a \preceq_p g(x)$.

Proof. (1) $x \preceq_r a$ iff $x \preceq_{p,q} a$ or $x \prec^{R1p} a$ and (1) holds in both cases.
(2) $a \preceq_r x$ iff $a \preceq_{p,q} x$ or $a \prec^{R3p} x$ or $a \prec^{R4p} x$ or $a \prec^{R4q} x$, and (2) holds in every case. □

Claim 2.24. *If $x \preceq_r y$ then $g(x) \preceq_p g(y)$ for $x, y \in A_r$.*

Proof. $x \prec_r y$ iff $x \prec_{p,q} y$ or $x \prec^{R1p} y$ or $x \prec^{R1q} y$ or $x \prec^{R2} y$ or $x \prec^{R3p} y$ or $x \prec^{R3q} y$ or $x \prec^{R4p} y$ or $x \prec^{R4q} y$ or $x \prec^{R5p} y$ or $x \prec^{R5q} y$, and the implication holds in every case. □

Claim 2.25. *If $v \preceq_p g(s)$ then $y_v \preceq_r s$ for $v \in A_p \setminus A$ and $s \in A_r$.*

Proof. If $s \in A_p$ ($s \in A_q$) then $g(s) = s$ ($g(s) = s'$) and so $y_v \prec^{R1p} s$ ($y_v \prec^{R1q} s$).

If $s = y_x$ for some $x \in A_p$ then $g(s) = x$ and so $y_v \preceq^{R2} y_x$.

If $s = u_x$ for some $x \in L^+$ then $y_v \preceq_r y_x$, and so $y_v \prec^{R4p} u_x$. □

Claim 2.26. *If $x \preceq_r y$ and $\delta_{g(x)} < \delta_{g(y)}$ then there is $a \in A$ such that $x \preceq_r a \preceq_r y$.*

Proof. By Claim 2.24 we have $g(x) \preceq_p g(y)$. Hence, by Claim 2.13, there is $a \in A$ such that $g(x) \preceq_p a \preceq_p g(y)$. Then, by Claim 2.23, we have $x \preceq_r a \preceq_r y$. □

Claim 2.27. *If $a \in A$ and $x \in A_r$, $a \preceq_r x$, then $\pi(a) \in \mathrm{o}^*(x)$ iff $\pi(a) \in \mathrm{o}^*(g(x))$.*

Proof. We can assume that $x \notin A_p \cup A_q$. If $x \in Y$ then Claim 2.17 implies the statement. If $x = u_z$ for some $z \in L^+$ then $g(x) = z$, $\pi(a) < \delta_z$ and $\mathrm{o}^*(z) \cap \delta_z = \mathrm{o}^*(u_z) \cap \delta_z = o_B(\pi_B(z))$, and so we are done. □

Claim 2.28. *If $x \in A_r \setminus A$, $v \in A_p \setminus A$, $v \prec_p g(x)$ and $\delta_v = \delta_{g(x)}$ then $\pi(y_v) \in \mathrm{o}^*(x)$.*

Proof. We have $\pi(y_v) = \beta_v \in \overline{\mathrm{o}}(\delta_v) \cap [\underline{\gamma}(\delta_v), \gamma(\delta_v))$. If $x \in (A_p \cup A_q) \setminus A$, then $\beta_v \in \mathrm{o}^*(x)$ by Claim 2.16.

If $x = y_z$ for some $z \in A_p$, we have $z = g(x)$ and then $\beta_v \in \mathrm{o}^*(y_z)$ by Claim 2.17.

If $x = u_z$ for some $z \in L^+$ then $\beta_v \in \mathrm{o}^*(z)$ because p is good. Now as $\beta_v < \delta_z$ and $\mathrm{o}^*(z) \cap \delta_z = \mathrm{o}^*(u_z) \cap \delta_z$, the statement holds. □

Claim 2.29. *If $s \in A_r \setminus (A \cup Y)$ and $v = g(s)$ then $\pi(y_v) \in \mathrm{o}^*(s)$.*

Proof. We have $\pi(y_v) = \beta_v \in \overline{\mathrm{o}}(\delta_v) \cap \gamma(\delta_v)$. If $s \in A_p \cup A_q$ then $\overline{\mathrm{o}}(\delta_v) \cap \gamma(\delta_v) \subset \mathrm{o}^*(s)$ because p and q are good. If $s = u_{g(s)}$ then the block orbit of s and the block orbit of $g(s)$ are the same and the block orbit of $g(s)$ contains $\overline{\mathrm{o}}(\delta_v) \cap \gamma(\delta_v)$ because p is good. □

Claim 2.30. *If $w \in A_p$, $s \in A_r$, $w \preceq_r s$ and $\delta_w = \delta_{g(s)}$ then $s \in A_p$.*

Proof. If $s \in A_q \setminus A_p$ then $w \preceq_{p,q} s$ and so there is $a \in A$ such that $w \preceq_p a \preceq_q s$ which contradicts $\delta_w = \delta_{g(s)}$.

If $s = y_{g(s)}$ then $w \prec^{R3p} s$, i.e. there is $a \in A$ with $w \preceq_p a \preceq_p g(s)$ which contradicts $\delta_w = \delta_{g(s)}$.

If $s = u_{g(s)}$ then $w \prec^{R4p} u_{g(s)}$, i.e. $w \preceq_r y_{g(s)}$, but this was excluded in the previous paragraph. □

Lemma 2.31. *There is a function $\mathrm{i}_r \supset \mathrm{i}_p \cup \mathrm{i}_q$ such that $\langle A_r, \preceq_r, \mathrm{i}_r \rangle$ satisfies (P4) and (P5).*

Proof. If $\{s,t\} \in \left[A_p\right]^2$ ($\{s,t\} \in \left[A_q\right]^2$) we will have $\mathrm{i}_r\{s,t\} = \mathrm{i}_p\{s,t\}$ ($\mathrm{i}_r\{s,t\} = \mathrm{i}_q\{s,t\}$), and so (P5) holds because p and q satisfy (P5).

To check (P4) we should prove that $\mathrm{i}_p\{s,t\}$ is the greatest common lower bound of s and t in $(A_r, \preceq_r)$.

Indeed, let $x \preceq_r s, t$. We can assume that $x \notin A_p$. Then, we distinguish the following three cases.

Case i. $x \in A_q \setminus A_p$.

Then there are $a, b \in A$ such that $x \preceq_q a \preceq_p s$ and $x \preceq_q b \preceq_p t$. Thus $x \preceq_q \mathrm{i}_q\{a,b\} = \mathrm{i}_p\{a,b\} \preceq_p \mathrm{i}_p\{s,t\}$ and so $x \preceq_{p,q} \mathrm{i}_p\{s,t\}$.

Case ii. $x \in Y$.

Then $x \prec^{R1p} s$ and $x \prec^{R1p} t$, i.e. $g(x) \preceq_p s$ and $g(x) \preceq_p t$. So $g(x) \preceq_p \mathrm{i}_p\{s,t\}$ and hence $x \prec^{R1p} \mathrm{i}_p\{s,t\}$.

Case iii. $x \in U$.

Put $x = u_z$ for some $z \in L^+$. Since $x \preceq_r s, t$, we have that $u_z \prec^{R5p} s, t$, and thus $z \preceq_p s, t$. So $z \preceq_p i_p\{s,t\}$, and hence $x \preceq_r i_p\{s,t\}$.

Assume now that $s, t \in A_r$ are $\preceq_r$-compatible, but $\preceq_r$-incomparable elements, $\{s,t\} \notin \left[A_p\right]^2 \cup \left[A_q\right]^2$. Write $v = \mathrm{i}_p\{g(s), g(t)\}$. Note that, by Claim 2.24, $g(s)$ and $g(t)$ are compatible in p and hence $v \in A_p$. Let

$$\mathrm{i}_r\{s,t\} = \begin{cases} v & \text{if } v \in A, \\ y_v & \text{otherwise.} \end{cases}$$

Case I. $v \in A$.

Then $g(s)$ and $g(t)$ are incomparable in A_p. Indeed, $g(s) \preceq_p g(t)$ implies $v = g(s)$ and so $s = g(s) \preceq_r t$ by Claim 2.23.

Thus $\pi(v) \in \mathrm{o}^*(g(s)) \cap \mathrm{o}^*(g(t))$ by applying (P5) in p. Note that $v \preceq_r s, t$ by Claim 2.23. So, $\pi(v) \in \mathrm{o}^*(s) \cap \mathrm{o}^*(t)$ by Claim 2.27. Hence (P5) holds.

We have to check that v is the greatest lower bound of s, t in $(A_r, \preceq_r)$. We have $v \preceq_r s, t$ by Claim 2.23.

Let $w \in A_r$, $w \preceq_r s, t$. Then $g(w) \preceq_p g(s), g(t)$ by Claim 2.24. So $g(w) \preceq_p v$. Then $w \preceq_r v$ by Claim 2.23.

Case II. *$v \notin A$.*

Then $\delta_{g(s)} = \delta_{g(t)} = \delta_v$ by Claim 2.23 and Claim 2.13 if $g(s)$ and $g(t)$ are comparable in A_p, and by Claim 2.15 if $g(s)$ and $g(t)$ are incomparable in A_p.

If $g(s)$ and $g(t)$ are incomparable in A_p then $v \prec_p g(s), g(t)$ and $s, t \notin A$ by Claim 2.14. So, $\pi(y_v) \in \mathrm{o}^*(s) \cap \mathrm{o}^*(t)$ by Claim 2.28.

If $g(s) \prec_p g(t)$ then $s \notin Y$ by Claim 2.25 and $s \notin A$ because $v = g(s) \notin A$. Then $\pi(y_v) \in \mathrm{o}^*(s)$ by Claim 2.29. Also, since $v = g(s) \prec_p g(t)$ we infer from Claim 2.23 that $t \notin A$ and so we have that $\pi(y_v) \in \mathrm{o}^*(t)$ by Claim 2.28. Hence (P5) holds.

We have to check that y_v is the greatest common lower bound of s, t in $(A_r, \preceq_r)$. First observe that $y_v \preceq_r s, t$ by Claim 2.25.

Let $w \preceq_r s, t$.

Assume first that $\delta_{g(w)} < \delta_v$. Then there are $a, b \in A$ with $w \preceq_r a \preceq_r s$ and $w \preceq_r b \preceq_r t$ by Claim 2.26 and so $g(w) \preceq_p \mathrm{i}_p\{a, b\} \preceq_p v$ by using Claim 2.23. Now since $g(y_v) = v$, we obtain $w \preceq_r \mathrm{i}_p\{a, b\} \preceq_r y_v$ again by Claim 2.23.

Assume now that $\delta_{g(w)} = \delta_v$. Since $\{s, t\} \notin [A_p]^2 \cup [A_q]^2$, we have that $w \notin U \cup U'$. Then, by Claim 2.30, $w = y_z$ for some $z \in A_p$. Then $z \preceq_p g(s)$ and $z \preceq_p g(t)$ by Claim 2.24, and so $z \preceq_p v$. Thus $y_z \preceq_r y_v$. $\square$

Now our aim is to verify condition (P6). First, we need some preparations.

For every $x, y \in A_r$ with $x \preceq_r y$ let

$$\pi_x(y) = \begin{cases} \pi(y) & \text{if } \pi_B(x) = \pi_B(y), \\ \pi_-(y) & \text{if } \pi_B(x) \neq \pi_B(y). \end{cases}$$

Note that for every $x, y \in A_r$ with $x \preceq_r y$, an interval $\Lambda \in \mathbb{I}$ isolates x from y iff $\Lambda^- < \pi(x) < \Lambda^+ \leq \pi_x(y)$.

Claim 2.32. *Let $a \in A$ and $t \in A_r$, $a \preceq_r t$. If Λ isolates a from t then Λ isolates a from $g(t)$.*

Proof. The statement is obvious if $t \in A_p$. Assume that $t \in A_q \setminus A_p$. Note that since Λ contains an element of A, we have that $\Lambda^+ \in Z$. Now if $t \in D \cup F \cup M$ we have that $Z \cap \pi(t) = Z \cap \pi(g(t)) = Z \cap \gamma(\delta_t)$, and so we are done. If $t \in L$ then as $a \preceq_r t$ we infer that $\pi_B(a) \neq \pi_B(t)$ and $\pi(a) < \delta_t = \pi_-(t)$, hence we have $\pi(a) < \Lambda^+ \leq \pi_a(t) = \pi_a(g(t)) = \pi_-(t)$, and so the statement holds.

If $t = y_v$ for some $v \in A_p$, then $a \prec_p v = g(t)$ and $\pi_a(y_v) \leq \pi_a(v)$, and so we are done.

If $t = u_v$ for some $v \in L^+$, we have $a \prec_p v = g(t)$ and $\pi_a(u_v) = \pi_a(v) = \pi_-(v)$. □

Claim 2.33. *Let $a \in A$ and $x \in A_r \setminus (A_p \cup A_q)$, $x \preceq_r a$. If Λ isolates x from a then $x = y_{g(x)}$ and Λ isolates $g(x)$ from a.*

Proof. We have $g(x) \preceq_p a$ by Claim 2.23, so as $a \in A$ we infer that $g(x) \notin L \cup M$, and thus $x \notin U \cup U'$. Hence $x \in Y$ and $g(x) \in D \cup F$, and so $x = y_{g(x)}$ and $\pi(g(x)) < \delta_{g(x)}$.

Let $J(\delta_{g(x)}) = \mathrm{I}(\pi(g(x)), j)$ and $\Lambda = \mathrm{I}(\pi(x), \ell)$. If $\ell > j$ then $\Lambda^- = \pi(y_{g(x)}) = \pi(x)$, which is impossible. If $\ell \leq j$ then $J(\delta_{g(x)}) \subset \Lambda$ and so $\Lambda^- < \pi(g(x)) < \Lambda^+$, i.e. Λ isolates $g(x)$ from a. □

Lemma 2.34. *$(A_r, \preceq_r, i_r)$ satisfies (P6).*

Proof. Assume that $\{s, t\} \in \left[A_r\right]^2$, $s \preceq_r t$ and Λ isolates s from t. Suppose that $\pi(s) \neq \pi_-(s)$ if $s \notin B_S$. So, $s \notin U \cup U'$. We should find $v \in A_r$ such that $s \preceq_r v \preceq_r t$ and $\pi(v) = \Lambda^+$. Note that since $s \preceq_r t$, we have $\delta_{g(s)} \leq \delta_{g(t)}$ by Claims 2.24 and 2.12.

We can assume that $\{s, t\} \notin \left[A_p\right]^2 \cup \left[A_q\right]^2$ because p and q satisfy (P6).

Case 1. $\delta_{g(s)} < \delta_{g(t)}$.

By Claim 2.26 there is $a \in A$ with $s \preceq_r a \preceq_r t$. Moreover, $g(s) \preceq_p a \preceq_p g(t)$ by Claim 2.23.

Case 1.1. $\pi(a) \in \Lambda$.

Then $\pi_B(s) = \pi_B(a)$ and so $\pi_s(t) = \pi_a(t)$. Thus Λ isolates a from t.

If $t \in A_p$ ($t \in A_q$) then applying (P6) in p (in q) for a, t and Λ we obtain $b \in A_p$ ($b \in A_q$) such that $a \preceq_p b \preceq_p t$ ($a \preceq_q b \preceq_q t$) and $\pi(b) = \Lambda^+$. Then $s \preceq_r a \preceq_{p,q} b \preceq_{p,q} t$, so we are done.

Assume now that $t \notin A_p \cup A_q$.

By Claim 2.32, the interval Λ isolates a from $g(t)$. Since $\pi_-(a) \neq \pi(a)$ if $a \notin B_S$, we can apply (P6) in p to get a $b \in A_p$ with $\pi(b) = \Lambda^+$ and $a \preceq_p b \preceq_p g(t)$.

Note that as $\pi(a) \in \Lambda, a \in A$ and $\pi(b) = \Lambda^+$, we have that $\pi(b) \in Z$.

If $\pi_B(a) = \pi_B(b)$, we have $b \notin M \cup L$ because $a \in A$.

If $\pi_B(a) \neq \pi_B(b)$, then $\pi_-(b) = \pi(b) = \Lambda^+ \leq \pi(t)$. Note that if $t \in U \cup U'$, then $\pi(t) = \Lambda^+$, and so we are done. Thus, we may assume that $t \in Y$. Then, we have $\pi_B(b) = \pi_B(t) = \pi_B(g(t))$ and $g(t) \in F$. Hence $b \in K \cup F$.

In both cases we have $b \notin M \cup L$, so $\pi(b) \in Z$ implies $b \in A$. Thus $b \preceq_r t$ by Claim 2.23, and so b witnesses (P6).

Case 1.2. $\pi(a) \notin \Lambda$.

Since p and q satisfy $(P6)$ and Λ isolates s from a, we can assume that $s \notin A_p \cup A_q$.

Hence $s = y_{g(s)}$ and Λ isolates $g(s)$ from a by Claim 2.33. Since $\pi(g(s)) \neq \pi_-(g(s))$ if $g(s) \notin B_S$, there is $v \in A_p$ with $g(s) \preceq_p v \preceq_p a$ and $\pi(v) = \Lambda^+$. Since $y_{g(s)} \preceq_r g(s)$ by the definition of $\preceq_r$, we have that v witnesses (P6).

Case 2. $\delta_{g(s)} = \delta_{g(t)}$.

Case 2.1. $s \in A_p$.

Since $s \in A_p$, $s \preceq_r t$ and $\delta_s = \delta_{g(t)}$ we infer from Claim 2.30 that $t \in A_p$, which was excluded.

By means of a similar argument, we can show that $s \in A_q$ is also impossible.

Case 2.2. $s = y_v$ *for some* $v \in A_p$.

We have that $\delta_v = \delta_{g(t)}$. Note that since $\Lambda^- < \pi(s) < \Lambda^+$, we have $\delta_v \leq \Lambda^+$.

Thus $\pi(t) \geq \Lambda^+ \geq \delta_v = \delta_{g(t)}$. Since we can assume that $\pi(t) > \Lambda^+$, we have $\pi(t) > \delta_{g(t)}$. If $t \in A_p \cup A_q$ and $g(t) \in F \cup D \cup M$, or $t \in Y$, or $t \in U \cup U'$ then $\pi(t) \leq \delta_{g(t)}$. Thus we have $t \in A_p \cup A_q$ and $g(t) \in L$.

Note that as $\pi_B(t) \neq S$, if $\pi_B(y_v) = \pi_B(t)$ we would infer that $v \in F$ and hence $\delta_t = \delta_{g(t)} < \delta_v$. So $\pi_B(s) \neq \pi_B(t)$. Now since Λ isolates s from t, we deduce that $\delta_v = \delta_t = \Lambda^+$, and hence $\Lambda = \mathrm{J}(\delta_t)$.

Assume that $t \in A_q$ (the case $t \in A_p$ is simpler). Then $g(t) = t' \in L$. Since $\pi(t) > \delta_t = \pi_-(t)$ we have $\pi(t') > \pi_-(t')$ and so $t' \in L^+$.

Since $y_v \preceq_r t$ we have $y_v \prec^{R1q} t$, i.e. $v \preceq_p t'$ and so $y_v \preceq^{R2} y_{t'}$. Thus $y_v \prec^{R4q} u_t$. Hence $y_v \preceq_r u_t \preceq_r t$ and $\pi(u_t) = \delta_t = \Lambda^+$, i.e. u_t witnesses that (P6) holds. $\square$

This completes the proof of Lemma 2.5, i.e. $\mathcal{P}$ satisfies κ^+-c.c. $\square$

References

[1] I. Juhász, S. Shelah, L. Soukup, Z. Szentmiklóssy, *A tall space with a small bottom.* Proc. Amer. Math. Soc. 131 (2003), no. 6, 1907–1916.

[2] I. Juhász, L. Soukup, W. Weiss *Cardinal Sequences of length* $< \omega_2$ *under GCH* Fund. Math. 189 (2006), No.1, 35-52.

[3] I. Juhász, W. Weiss *On thin-tall scattered spaces.* Colloq. Math. 40 (1978/79), no. 1, 63–68

[4] I. Juhász, W. Weiss, *Cardinal sequences.* Ann. Pure Appl. Logic 144 (2006), no. 1-3, 96–106.

[5] J. C. Martínez, *A forcing construction of thin-tall Boolean algebras.* Fund. Math. 159 (1999), no. 2, 99–113.

[6] J. C. Martínez, L. Soukup *Universal locally compact scattered spaces*, in preparation.

Facultat de Matemàtiques, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain

E-mail address: jcmartinez@ub.edu

Alfréd Rényi Institute of Mathematics

E-mail address: soukup@renyi.hu